Applications of the theta function to sums of squares

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Abstract. Consider the number of representations of a number as a sum of 2, 4, 6 or 8 squares. Jacobi proved formulas expressing the number of such representations as a sum of squares in terms of sums over divisors. This paper reviews these results and the derivations in the case of sums of 2 and 4 squares using the theory of theta functions and modular functions, where changes in sign and permutations of the squares count as distinct representations. Presented is an introduction to modular functions, leading to the result that all bounded modular functions are constant; which is used to prove that the appropriate even powers of theta functions are their so-called Lambert series. By writing the nth power of the theta function as the generating function corresponding to the sequence given by number of representations of a nonnegative integer as a sum of n squares, the formulas arise by comparing coefficients. The formula for representations as a sum of 4 squares yields Lagrange's four squares theorem as a consequence, that is, every nonnegative integer is a sum of 4 squares.

1 Introduction

Consider the problem of whether a given positive integer can be expressed as the sum of N squares, and if so, in what number of ways, for some positive integer N. If $N = 4$, for example, the integer 60 can be expressed as a sum of 4 squares as

$$60 = 1^2 + 1^2 + 3^2 + 7^2 = 1^2 + 3^2 + 5^2 + 5^2 = 2^2 + 2^2 + 4^2 + 6^2,$$

and we can check that these three expansions are the only ways to represent 60 as a sum of four squares, up to permuting the squares or changing the signs (from x^2 to $(-x)^2$). If we do count two representations of a number as a sum of four squares which can be obtained from each other by permuting the squares or changing the signs as described above, then each of the three representations of 60 actually correspond to $2^4 \cdot 12 = 192$ representations, giving a total of 576 ways to represent 60 as a sum of four squares. It turns out that we can count the number of representations of a positive integer as a sum of four squares, in the sense that we can determine the size of the set $\{(a, b, c, d) \in \mathbb{Z}^4 \mid a^2 + b^2 + c^2 + d^2 = n\}$, by the use of theta functions.

We denote the number of representations of a positive integer n as a sum of k squares as (see [2, 3])

$$r_k(n) = \left|\{(a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k \mid a_1^2 + a_2^2 + \ldots + a_k^2 = n\}\right|.$$

In our example above, we computed that $r_4(60) = 576$. To compute the coefficients $r_k(n)$, we will consider the associated generating function in n, and relate this to the Jacobi theta function, given by

$$\theta(\tau) = \sum_{a\in\mathbb{Z}} e^{\pi i\tau a^2}. \tag{1}$$

In particular, we can expand the function $\theta(\tau)^k$ as (this is worked out in the case $n = 2$ in [3])

$$\theta(\tau)^k = \Big(\sum_{a\in\mathbb{Z}} e^{\pi i\tau a^2}\Big)^k = \sum_{a_1,a_2,\ldots,a_k\in\mathbb{Z}} e^{\pi i\tau(a_1^2+a_2^2+\ldots+a_k^2)} = \sum_{n=0}^{\infty} r_k(n)e^{\pi i\tau n}, \tag{2}$$

so that this is the generating function for the coefficients $\{r_k(n)\}$ in the variable $e^{\pi i\tau}$. In the last equality we rewrote the sum over the a_i's as a sum over $n = a_1^2 + a_2^2 + \ldots + a_k^2$ and over $\{(a_1, a_2, \ldots, a_k) \in \mathbb{Z}^k \mid a_1^2 + a_2^2 + \ldots + a_k^2 = n\}$; we may perform this change of summation since the defining sum for the theta function is absolutely convergent if $\tau \in \mathbb{H}$, that is, $\operatorname{Im}\tau > 0$.

Using the generating function expression, we will show, following [3], that the following formulae hold for $r_2(n)$ and $r_4(n)$.

Theorem 1. (Jacobi) [3] *Let n be any positive integer, then*

$$r_2(n) = \sum_{d|n, d\equiv 1 \pmod 4} 4 \quad - \sum_{d|n, d\equiv 3 \pmod 4} 4$$

and

$$r_4(n) = \sum_{d|n, 4\nmid d} 8d.$$

The methods can be expanded (see Chapter 9 of [2] and a problem in Chapter 10 of [3]) to prove the similar formulae for $k = 6$ and 8 below.

Theorem 2. (Jacobi) [2, 3] *Let n be any positive integer, then*

$$r_6(n) = \sum_{d|n, d\equiv 1 \pmod 4} 4\left(4d^2 - \frac{n^2}{d^2}\right) - \sum_{d|n, d\equiv 3 \pmod 4} 4\left(4d^2 - \frac{n^2}{d^2}\right)$$

and

$$r_8(n) = \sum_{d|n} 16(-1)^{n+d} d^3.$$

We will prove the first two formulas in Section 5. The method we will use (following [2, 3]) is to consider the generating functions associated to both sides of the equations in Theorems 1 and 2, and show that these generating functions are equal by proving that they have the same limiting behavior for complex numbers $e^{i\pi\tau}$ with large magnitude, and using the maximum modulus principle.

2 Modular Functions

We now will consider a class of functions with given transformation properties (entire modular functions) and prove that all such functions are constant. The aim is to construct a series expansion for the kth power of the theta function $\theta(\tau)^k$, and show that this series expansion is equal to $\theta(\tau)^k$ by showing that the ratio of these is an entire modular function. Before we define these functions, we will consider the properties of what is known as the modular group acting on the upper half plane $\mathbb{H}$. Define the modular group Γ (see [2, 3]) to be the group of automorphisms of the upper half plane $\mathbb{H}$, generated by $\phi(\tau) = \tau + 1$ and $\psi(\tau) = -1/\tau$. We can see that by repeated composition of the functions ϕ, ϕ^{-1} and ψ, the modular group consists of all linear fractional transformations $f(\tau) = \frac{a\tau+b}{c\tau+d}$ which have integer coefficients a, b, c and d, and with determinant $ad - bc = 1$. Indeed, by applying the Euclidean algorithm to $a\tau + b$ and $c\tau + d$, we obtain a sequence of automorphisms ϕ^r and ψ whose composition is $\frac{a\tau+b}{c\tau+d}$. We can show that every linear fractional transformation $f(\tau) = \frac{a\tau+b}{c\tau+d}$ with determinant 1 is in the modular group Γ by induction on $|c|$. The base case, $c = 0$, follows from the fact that $ad = 1$ and hence without loss of generality $a = d = 1$, so that $\frac{a\tau+b}{c\tau+d} = \phi^b(\tau)$. For $|c| > 0$, we may assume without loss of generality that $c > 0$, and by translation by an integer it suffices to prove this for $0 \le a \le c - 1$. Note that the composition of two linear fractional transformations with determinant 1 also has determinant 1, and ϕ and ψ have determinant 1. Then, by the inductive hypothesis, $-1/f(\tau)$ has denominator $a\tau + b$ with $a \le c - 1$, is contained in $\langle\phi, \psi\rangle$, and has determinant 1, hence $f \in \langle\phi, \psi\rangle$ as desired. For example, the Euclidean algorithm procedure enables us to compute that

$$\frac{4\tau+9}{3\tau+7} = \phi\Big(\frac{\tau+2}{3\tau+7}\Big) = (\phi\circ\psi)\Big(-\frac{3\tau+7}{\tau+2}\Big) = (\phi\circ\psi\circ\phi^{-3})\Big(-\frac{1}{\tau+2}\Big) = (\phi\circ\psi\circ\phi^{-3}\circ\psi\circ\phi^2)(\tau),$$

where after the first two steps, we reduced the magnitude of the coefficient of τ in the denominator. Finally, consider the relation

$$\operatorname{Im}\frac{a\tau+b}{c\tau+d} = \frac{\operatorname{Im}((a\tau+b)(c\overline{\tau}+d))}{|c\tau+d|^2} = \frac{(ad-bc)\operatorname{Im}\tau}{|c\tau+d|^2} = \frac{\operatorname{Im}\tau}{|c\tau+d|^2}, \tag{3}$$

which holds since $ad-bc = 1$. We see that all linear fractional transformations with positive determinant are automorphisms of the upper half plane $\mathbb{H}$. Thus, the modular group Γ acts on $\mathbb{H}$ by special linear fractional transformations.

The fundamental domain $\mathfrak{F}$ is defined as the set of all complex numbers $\tau \in \mathbb{H}$ with $-1/2 \le \operatorname{Re}\tau \le 1/2$ and $|\tau| \ge 1$; we claim that every orbit in $\mathbb{H}$ with respect to the action by Γ intersects $\mathfrak{F}$ (the proof presented is from [1, 3]). Consider the orbit of a given $\tau \in \mathbb{H}$ under the action of the modular group Γ. We first show that the set $\{r\tau + s \mid (r, s) \in \mathbb{Z}^2\}$ attains its minimum. If $|r\tau + s| \le |\tau|$, we must have $\operatorname{Im}(r\tau + s) = r\operatorname{Im}\tau \le |\tau|$, and

$-|\tau| \leq s \leq |\tau| + \frac{|\tau| \operatorname{Re} \tau}{\operatorname{Im} \tau}$, so that there are finitely many possible pairs (r, s) with this property. Since the number of such pairs is nonzero (as $(r, s) = (1, 0)$ satisfies $|r\tau + s| \leq |\tau|$), we may choose (r, s) such that $|r\tau + s|$ is minimal.

If $f(\tau) = \frac{a\tau+b}{r\tau+s}$ is in Γ, using equation (3), we have that $\operatorname{Im} f(\tau)$ is maximal in the set of all $\operatorname{Im} \tau'$, for τ' in the orbit of τ. In particular, if $\tau' = -1/f(\tau)$, by equation (3),

$$\operatorname{Im} f(\tau) \geq -\operatorname{Im} \frac{1}{f(\tau)} = \frac{\operatorname{Im} f(\tau)}{|r\tau + s|^2},$$

and $\left|f(\tau)\right| \geq 1$. However, we still must show that such an f exists. Note that $\gcd(r, s) = 1$, otherwise $\operatorname{Im}(\frac{r}{\gcd(r,s)}\tau + \frac{s}{\gcd(r,s)}) < \operatorname{Im}(r\tau+s)$. If $r = 0$, then f is a translation; and if $r = 1$, we can define $f(\tau) = \frac{\tau+(s-1)}{r\tau+s}$. Otherwise $r \neq 0, 1$, and we may define $0 \leq a \leq r - 1$ such that $a \equiv s^{-1} \pmod{r}$, and choose $b \in \mathbb{Z}$ such that $as = br + 1$, the linear fractional transformation $f_0(\tau) = \frac{a\tau+b}{r\tau+s}$ then has determinant 1. Choose some integer k such that $\left|\operatorname{Re} f_0(\tau) + k\right| \leq 1/2$. Then $f(\tau) := (\phi^k \circ f_0)(\tau)$ has $\left|\operatorname{Re} f(\tau)\right| < 1/2$ and by the earlier analysis, $\left|f(\tau)\right| \geq 1$, so that $f(\tau) \in \mathfrak{F}$ is in the fundamental domain. Thus the orbit of τ with respect to the group action by Γ on $\mathbb{H}$ intersects the fundamental domain $\mathfrak{F}$ at the point $f(\tau)$. The results obtained thus far about the modular group are summarized in the following lemma.

Lemma 3. [1, 3] *The modular group Γ of linear fractional transformations $f(\tau) = \frac{a\tau+b}{c\tau+d}$ acting on the upper half-plane $\mathbb{H}$ is generated by the translation $\phi(\tau) = \tau+1$ and the automorphism $\psi(\tau) = -1/\tau$, and every orbit in $\mathbb{H}$ corresponding to the action by Γ has a nonempty intersection with the fundamental domain $\mathfrak{F} := \{\tau \in \mathbb{C} \mid |\operatorname{Re} \tau| \leq 1/2, |\tau| \geq 1\}$.*

Let $G \leq \Gamma$ be a finite-index subgroup of the modular group. Define (as in [2], Chapter 11) a *modular function* for G as a meromorphic function $h : \mathbb{H} \to \mathbb{C}$ in the variable τ with the transformation property $h(g(\tau)) = h(\tau)$ for all $g \in G$. We follow [2] in defining an entire modular function. Since G has finite index, we may choose a union $\mathfrak{F}_G$ of finitely many sets of the form $f(\mathfrak{F})$ where f is a linear fractional transformation of determinant 1, such that every orbit in $\mathbb{H}$ with respect to the action by G has nonempty intersection with $\mathfrak{F}_G$. In particular, we may consider representatives $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Gamma$ such that Γ is the disjoint union of the right cosets $G\alpha_i$ for $1 \leq i \leq n$. Then every orbit in $\mathbb{H}$ with respect to the action by G has a nonempty intersection with the union $\mathfrak{F}_G := \alpha_1(\mathfrak{F}) \cup \alpha_2(\mathfrak{F}) \cup \ldots \cup \alpha_n(\mathfrak{F})$, by Lemma 3. Note that the closure of the fundamental domain $\mathfrak{F} \subseteq S^2$ in the Riemann sphere also includes the point ∞. By applying the automorphism α_i of S^2, we see that the closure of $\alpha_i(\mathfrak{F})$ is the union of $\alpha_i(\mathfrak{F})$ with $\alpha_i(\infty)$. Again applying the fact that G has finite index, the intersection $G \cap \langle\phi\rangle$ must be nontrivial, and there exists some n such that $\phi^n \in G$. For the point ∞, we may define a function g_∞ on the punctured unit disk by $g_\infty(q_\infty) = f(\tau)$, where $q_\infty = e^{2\pi i\tau/n}$. For a point $\alpha \in \mathbb{Q}$, we may choose some $\chi_\alpha \in \Gamma$ such that $\chi_\alpha(\alpha) = \infty$. Since G has finite index, $G \cap \langle\chi_\alpha\phi\chi_\alpha^{-1}\rangle$ must be nontrivial (as $\langle\chi_\alpha\phi\chi_\alpha^{-1}\rangle$ is infinite), and there exists some n_α such that $\chi_\alpha\phi^{n_\alpha}\chi_\alpha^{-1} \in G$. We may define the function g_α, again on the punctured unit disk, such that $g_\alpha(q_\alpha) = f(\tau)$, where $q_\alpha := e^{2\pi i\chi_\alpha(\tau)/m}$. For the purpose of simplifying expressions, we will define $g_i := g_{\alpha_i(\infty)}$ and $q_i := q_{\alpha_i(\infty)}$.

We say, as in [2], that h is an *entire modular function* if g_i can be analytically continued to a holomorphic function in the open unit disk for all $1 \leq i \leq n$. We then obtain the following result (stated in [2], and proved in a particular case in Chapter 10 of [3]).

Lemma 4. [2, 3] *Let $G \leq \Gamma$ be a subgroup of finite index. Then the only entire modular functions on $\mathbb{H}$ with respect to G are the constant functions. Further, every bounded modular function on $\mathbb{H}$ with respect to G is constant.*

Proof. We follow the method of proof in [3]. We will be using the maximum modulus principle, which states that a holomorphic function on an open set will not attain its maximum, because it is an open mapping. Let h be an entire modular function on $\mathbb{H}$ with respect to G. In the notation of the preceding discussion, since h is entire, the function h can be analytically continued to a holomorphic function on the closure of $\mathfrak{F}_G$ in the Riemann sphere S^2. Since this closure is compact, by the extreme value theorem, h attains its maximum at some point in this closure. By the maximum modulus principle, h does not attain its maximum on $\mathbb{H}$, hence the maximum must be attained at one of the cusps $\alpha_i(\infty)$. But, we may apply the maximum modulus principle to g_i on the open unit disk to see that g_i does not attain its maximum at $q_i = 0$, a contradiction. By Riemann's theorem on removable singularities (see [3], Chapter 3), any bounded modular function is entire, so all bounded modular functions are thus constant. □

For the purposes of theta functions, due to the fact ([3], Chapter 10) that

$$\theta(\tau) = \theta(\phi^2(\tau)) \quad \text{and} \quad \theta^2(\tau) = \frac{i}{\tau}\theta^2(\psi(\tau)), \tag{4}$$

which will be derived in Section 3, we will consider the subgroup $G = \langle \phi^2, \psi \rangle$ of Γ. To show that all entire modular functions on G are constant, we are left to show that G has finite index in Γ. To this end, the following statement holds.

Lemma 5. *The subgroup $G := \langle \phi^2, \psi \rangle \leq \Gamma$ has index 3 in Γ, and in particular, $\Gamma = G \cup G\phi \cup G\phi\psi$.*

Proof. We will be using congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, which are considered in [1]. First, we may verify that $\Gamma \cong \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ under the identification

$$\left(f(\tau) = \frac{a\tau + b}{c\tau + d} \right) \in \Gamma \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}.$$

Define the *congruence subgroup* $\widetilde{\Gamma} \leq \Gamma$ to be the group consisting of all matrices $M \in \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ with either

$$M \equiv \mathbb{I} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2,$$

and define (as in [1], Chapter 3) the *principal congruence subgroup* $\Gamma(2)$ of level 2 as the group of all matrices $M \in \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ with $M \equiv \mathbb{I} \pmod 2$. We will use the Euclidean algorithm approach from earlier in the section to show that every element of $\widetilde{\Gamma}$ corresponds to an element of G. Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\Gamma},$$

and $f(\tau)$ be the associated linear fractional transformation; we show that $M \in G$ by induction on c, where we may multiply M by an element of $\{\pm 1\}$ such that $c \geq 0$. Note that the given condition can be rephrased as $a \equiv d \pmod 2$ and $b \equiv c \pmod 2$. If $c = 0$, $ad = 1$, and by multiplying by an element of $\{\pm 1\}$, we see that M is a translation in $\langle \phi^2 \rangle$, as b is even. Suppose that $c > 0$. As $\gcd(a, c) = 1$, there exists some integer k such that $|a + 2ck| \leq c - 1$. Then $(\psi \circ \phi^{2k} \circ f)(\tau) = -\frac{c\tau + d}{(a+2ck)\tau + (b+2dk)}$ is a linear fractional transformation corresponding to a matrix in $\widetilde{\Gamma}$, with $|a + 2ck| < c$, so by the inductive hypothesis, $\psi \circ \phi^{2k} \circ f \in G$ and $f \in G$ as well. To compute the index of G in $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$, it suffices to compute the index of $\langle \psi \rangle = \langle \phi^2, \psi \rangle$ in $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) = [\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}]/\Gamma(2)$ (as $\phi^2 \cong \mathbb{I} \pmod 2$). Note that $\left|\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})\right| = 6$, and $\left|\langle \psi \rangle\right| = \left|\{1, \psi\}\right| = 2$, hence G has index 3 in Γ. We can verify that G, $G\phi$, and $G\phi\psi$ correspond to disjoint cosets in $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, proving the lemma. □

3 Properties of the theta function

We will first consider the properties of the Jacobi theta function with respect to modular transformations, that is, with respect to elements of the modular group. Throughout this section, we will use the notation $G = \langle \phi^2, \psi \rangle \leq \Gamma$. The identity $\theta(\phi^2(\tau)) = \theta(\tau + 2) = \theta(\tau)$ follows immediately from the expression (1). To prove the other identity in equation (4), we use the Poisson summation formula to deduce the following lemma. Note that in all uses of the Poisson summation formula in this section and in the next, the relevant function has exponential decay at infinity.

Lemma 6. [2, 3] *For any $\tau \in \mathbb{H}$, we have the relation*

$$e^{-\frac{\pi i}{4}} \sqrt{\tau} \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2} = \sum_{n \in \mathbb{Z}} e^{-\frac{\pi i n^2}{\tau}},$$

where $\sqrt{\tau}$ is defined using the normal branch cut along the negative real line.

Proof. Following [3], we will apply the Poisson summation formula to the entire function $g(z) := e^{\pi i \tau z^2}$. In particular,

$$\begin{aligned}\sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2} &= \sum_{n \in \mathbb{Z}} g(n) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} g(x) e^{-2\pi i n x} dx = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{\pi i \tau x^2 - 2\pi i n x} dx \\ &= \sum_{n \in \mathbb{Z}} e^{-\frac{\pi i n^2}{\tau}} \int_{-\infty}^{\infty} e^{\pi i \tau (x - \frac{n}{\tau})^2} dx = \sum_{n \in \mathbb{Z}} e^{-\frac{\pi i n^2}{\tau}} \int_{-\infty}^{\infty} e^{\pi i \tau x^2} dx.\end{aligned}$$

Let $\tau = |\tau|e^{i\omega}$. By considering the function e^{-z^2} and integrating it on the contour which is a sector of radius R centered at 0 from the point R to $Re^{i(-\frac{\pi}{4}+\frac{\omega}{2})}$, we obtain

$$\int_0^{R\sqrt{\pi|\tau|}} e^{\pi i\tau x^2}dx = \sqrt{\frac{1}{\pi|\tau|}}e^{i(\frac{\pi}{4}-\frac{\omega}{2})}\int_0^R e^{x^2}dx$$

and hence, by setting $R \to \infty$ and plugging this into the previous equation,

$$\sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2} = \frac{1}{\sqrt{|\tau|}}e^{i(\frac{\pi}{4}-\frac{\omega}{2})}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi i n^2}{\tau}} = \tau^{-\frac{1}{2}}e^{\frac{\pi i}{4}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi i n^2}{\tau}}$$

as desired. □

The transformation of $\theta(\tau)$ under ψ given in equation (4) follows as a consequence of this identity. Since $\Gamma = G \cup G\phi \cup G\phi\psi$ by Lemma 5, we see that every orbit of $\mathbb{H}$ with respect to the action by G has a nonempty intersection with $\mathfrak{F} \cup \phi(\mathfrak{F}) \cup (\phi \circ \psi)(\mathfrak{F})$. To show that a function of the form $S(\tau)/\theta(\tau)$ is a bounded modular function with respect to G, we must show that it is invariant under the action of G, $\theta(\tau)$ is nonzero everywhere, and $S(\tau)/\theta(\tau)$ is bounded near the points on the boundary. The point at infinity is mapped to itself by ϕ and is mapped to 1 by $\phi \circ \psi$, so we must show that $S(\tau)/\theta(\tau)$ is bounded near 1 and ∞.

We will now consider estimates for $\theta(\tau)$ as $\tau \to 1$ and $\tau \to \infty$, the corresponding estimates for the series $S(\tau)$ will be considered in the next section. If $\tau \to \infty$, then $q := e^{\pi i\tau} \to 0$, and every term in the theta function series (1) tends to zero except the $n = 0$ term, so that $\theta(\tau) \to 1$. The following lemma (from [3]) addresses the case $\tau \to 1$.

Lemma 7. [3] *As* $\operatorname{Im}\tau \to \infty$, *we have the relation*

$$\theta\left(\frac{\tau-1}{\tau}\right) = 2\sqrt{\tau}e^{\frac{\pi i(\tau-1)}{4}} + O(|\tau|^{\frac{1}{2}}e^{\frac{9\pi i\tau}{4}}),$$

where as in Lemma 6, $\sqrt{\tau}$ *is defined using the standard branch cut of the complex plane, that is, along the negative real axis.*

Proof. Set $\tilde{\tau} = -1/\tau$. Then by the Poisson summation formula,

$$\begin{aligned}\theta(\tilde{\tau}+1) &= \sum_{n\in\mathbb{Z}} e^{\pi i(\tilde{\tau}+1)n^2} = \sum_{n\in\mathbb{Z}} e^{\pi i n}e^{\pi i\tilde{\tau}n^2} = \sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} e^{\pi i x+\pi i\tilde{\tau}x^2}e^{-2\pi i n x}dx \\ &= \sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} e^{\pi i\tilde{\tau}x^2-2\pi i(n-\frac{1}{2})x}dx.\end{aligned}$$

Following the proof of Lemma 6, where we replace n by $n-\frac{1}{2}$, we obtain

$$\theta(\tilde{\tau}+1) = \tilde{\tau}^{-\frac{1}{2}}e^{\frac{\pi i}{4}}\sum_{n\in\mathbb{Z}} e^{-\frac{\pi i(n-\frac{1}{2})^2}{\tilde{\tau}}} = -\sqrt{\tau}e^{\frac{\pi i}{4}}\sum_{n\in\mathbb{Z}} e^{\pi i\tau\left(n-\frac{1}{2}\right)^2} = -2\sqrt{\tau}e^{\frac{\pi i}{4}}\sum_{n=1}^{\infty}\left(e^{\frac{\pi i\tau}{4}}\right)^{(2n-1)^2}.$$

As $\operatorname{Im}\tau \to \infty$, $e^{\frac{\pi i}{4}} \to 0$, and we can use the leading term as an estimate. Since

$$\left|\sum_{n=2}^{\infty}\left(e^{\frac{\pi i\tau}{4}}\right)^{(2n-1)^2}\right| \le e^{\frac{9\pi i}{4}}\sum_{n=0}^{\infty}|e^{\pi i\tau}|^{4n} = e^{\frac{9\pi i}{4}}(1-|e^{\pi i\tau}|^4)^{-1} = O(e^{\frac{9\pi i}{4}}),$$

we obtain the desired estimate for $\theta(\tau)$. □

We will outline the proof that the theta function $\theta(\tau)$ is nonzero everywhere on $\mathbb{H}$, by writing the theta function in a product form where this follows directly.

Theorem 8. [3] *The relation*

$$\theta(\tau) = \prod_{n=1}^{\infty}(1-e^{2\pi i\tau n})\left(1+e^{\pi i\tau(2n-1)}\right)^2$$

holds for all $\tau \in \mathbb{H}$. *In particular,* $\theta(\tau)$ *is nonzero.*

Proof. This proof is from [3]. To prove this theorem we introduce two functions which are generalizations of the left and right-hand-sides of the desired relation. These are the general form of the theta function

$$\Theta(w|\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2+2\pi inw},$$

and following the notation in [3], the function

$$\Pi(w|\tau) := \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi iw})(1+q^{2n-1}e^{-2\pi iw}),$$

where $q := e^{\pi i\tau}$. We will be analyzing the properties of these functions as entire functions in the variable $w \in \mathbb{C}$. First, we define, following [3], $c(w|\tau) = \Theta(w|\tau)/\Pi(w|\tau)$. This is an entire function since the only zeros of $\Pi(w|\tau)$ are the simple zeros at w such that $2w = (2n-1)\tau + (2m-1)$ for some $m, n \in \mathbb{Z}$, and $\Theta(w|\tau)$ has zeros at these values of w as well. Under the transformation $w \to w+\tau$, the theta function and the product become

$$\Theta(w+\tau|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2+2\pi in(w+\tau)} = e^{-\pi i\tau-2\pi iw}\sum_{n\in\mathbb{Z}} e^{\pi i\tau(n+1)^2+2\pi i(n+1)w} = e^{-\pi i\tau-2\pi iw}\Theta(w|\tau)$$

and

$$\begin{aligned}
\Pi(w+\tau|\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i(w+\tau)})(1+q^{2n-1}e^{-2\pi i(w+\tau)}) \\
&= \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n+1}e^{2\pi iw})(1+q^{2n-3}e^{-2\pi iw}) \\
&= \prod_{n=1}^{\infty} 1-q^{2n}\prod_{n=2}^{\infty} 1+q^{2n-1}e^{2\pi iw}\prod_{n=0}^{\infty} 1+q^{2n-1}e^{-2\pi iw} \\
&= \frac{1+q^{-1}e^{-2\pi iw}}{1+qe^{2\pi iw}}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi iw})(1+q^{2n-1}e^{-2\pi iw}) \\
&= q^{-1}e^{-2\pi iw}\Theta(w|\tau) = e^{-\pi i\tau-2\pi iw}\Pi(w|\tau),
\end{aligned}$$

so that $c(w+\tau|\tau) = c(w|\tau)$. From the definition of the theta and product functions, $c(w+1|\tau) = c(w|\tau)$. Thus, as $c(w|\tau)$ is bounded on the parallelogram with vertices 0, 1, τ, and $\tau+1$, and we may translate any point in $\mathbb{C}$ by $m\tau+n$ for some $m, n \in \mathbb{Z}$ to the parallelogram, we see that $c(w|\tau)$ is a bounded entire function. Thus by Liouville's Theorem, $c(w|\tau)$ does not depend on w. (The preceding analysis, that any entire function $f(w)$ with $f(w) = f(w+1) = f(w+\tau)$ is from [3], Chapter 9.) Now we show that $c(w|\tau)$ is independent of τ. Since $q \to 0$ as $\operatorname{Im}\tau \to \infty$, we see that in this limit $\Theta(w|\tau) \to 1$ and $\Pi(w|\tau) \to 1$, so that $c(w|\tau) = 1$ for all τ, which proves the theorem. To do this, it suffices to show the relation $c(w|\tau) = c(w|4\tau)$, so that $c(w\tau) = \lim_{k\to\infty} c(w|4^k\tau) = 1$, as in the limit $k \to \infty$, $q \to 0$. This follows from computing $c(1/4|\tau)$ and $c(1/2|\tau)$, the details can be found in [3]. □

4 Lambert Series

In this section we compute the *Lambert series expansions* for $\theta^{2k}(\tau)$ for $k = 1, 2, 3$ and 4. These expansions are derived in [2] by the manipulation of sums, however, we will follow the method in [3], Chapter 10 and prove these using Lemmas 4 and 5, and the Poisson summation formula. In the two cases, $k = 1$ and $k = 2$, proven here we will follow the method in [3] (the proof of the Lambert series for $\theta^2(\tau)$ is given there), by showing that the ratio of the series expansion and the corresponding power of the theta function is a bounded modular function in $\mathbb{H}$ with respect to the group G, which is constant by Lemma 4, since G has finite index in Γ by Lemma 5. We first determine the transformation properties of the series under ψ. We then have the following theorem, where the formulas are from [2, 3] and (6) has been modified from that in [2].

Theorem 9. [2, 3] *Let $\tau \in \mathbb{H}$, and let $q := e^{\pi i \tau}$. Consider the series*

$$S_2(\tau) = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}, \tag{5}$$

$$S_4(\tau) = 1 + 8\sum_{n=1}^{\infty} \left(\frac{nq^n}{1-q^{2n}} + \frac{q^{2n}}{(1+q^{2n})^2} \right), \tag{6}$$

$$S_6(\tau) = 1 + 4\sum_{n=1}^{\infty} \left(\frac{(2n-1)^2(-1)^n q^{2n-1}}{1-q^{2n-1}} + \frac{4n^2 q^n}{1+q^{2n}} \right), \tag{7}$$

$$S_8(\tau) = 1 + 16\sum_{n=1}^{\infty} \frac{n^3(q^n + (-1)^n q^{2n})}{1-q^{2n}}. \tag{8}$$

These satisfy the relation $S_{2k}(\tau) = (\frac{i}{\tau})^k S_{2k}(-\frac{1}{\tau})$ for all $\tau \in \mathbb{H}$.

Proof. The relation is proved here for $S_2(\tau)$ and $S_4(\tau)$ using the method in [3]; proofs of the relations $\theta^{2k}(\tau) = S_{2k}(\tau)$ for $k = 3$ and 4 can be found in [2]. For the first series, we use the Poisson summation formula to rewrite

$$S_2(\tau) = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} = 2\sum_{n\in\mathbb{Z}} \frac{q^n}{1+q^{2n}} = 2\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{e^{\pi i \tau x - 2\pi i n x}}{1+e^{2\pi i \tau x}} dx.$$

As in [3], we use the rectangular contour with height $-\operatorname{Im}(1/\tau)$ and width R, with $R \to \infty$, to evaluate the integral. The integrand has a pole at $x = -1/2\tau$ in this region, and the integrals over the vertical lines $\pm R + ai$ for $0 \le a \le -\operatorname{Im}(1/\tau)$ tend to 0 as $R \to \infty$. The remaining contour integral is then

$$\int_{-i\,\mathrm{Im}\,\frac{1}{\tau}-\infty}^{-i\,\mathrm{Im}\,\frac{1}{\tau}+\infty} \frac{e^{\pi i \tau y - 2\pi i n y}}{1+e^{2\pi i \tau y}} dy = -e^{\frac{2\pi i n}{\tau}} \int_{-\infty}^{\infty} \frac{e^{\pi i \tau x - 2\pi i n x}}{1+e^{2\pi i \tau x}} dx,$$

by making the substitution $x = y + 1/\tau$. Since

$$\operatorname{Res}\left(\frac{e^{\pi i \tau x - 2\pi i n x}}{1+e^{2\pi i \tau x}}; -\frac{1}{2\tau} \right) = -ie^{\frac{\pi i n}{\tau}} \lim_{x \to -\frac{1}{2\tau}} \frac{x + \frac{1}{2\tau}}{1+e^{2\pi i \tau x}} = \frac{1}{2\pi\tau} e^{\frac{\pi i n}{\tau}},$$

we obtain

$$\int_{-\infty}^{\infty} \frac{e^{\pi i \tau x - 2\pi i n x}}{1+e^{2\pi i \tau x}} dx = \frac{i}{\tau} \frac{e^{\frac{\pi i n}{\tau}}}{1+e^{\frac{2\pi i n}{\tau}}}, \tag{9}$$

and we may evaluate the first series as

$$S_2(\tau) = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} = \frac{2i}{\tau} \sum_{n\in\mathbb{Z}} \frac{e^{\frac{\pi i n}{\tau}}}{1+e^{\frac{2\pi i n}{\tau}}} = \frac{i}{\tau} \left(1 + 4\sum_{n=1}^{\infty} \frac{e^{-\frac{\pi i n}{\tau}}}{1+e^{-\frac{2\pi i n}{\tau}}} \right) = \frac{i}{\tau} S_2(-\frac{1}{\tau}).$$

For the second series, we define the function

$$f_\tau(x) = \frac{xe^{\pi i \tau x}}{1-e^{2\pi i \tau x}} + \frac{e^{2\pi i \tau x}}{(1+e^{2\pi i \tau x})^2} \tag{10}$$

away from $x = 0$, and at $x = 0$ we set

$$f_\tau(0) := \lim_{x\to 0} \left(\frac{xe^{\pi i \tau x}}{1-e^{2\pi i \tau x}} + \frac{e^{2\pi i \tau x}}{(1+e^{2\pi i \tau x})^2} \right) = -\frac{1}{2\pi i \tau} + \frac{1}{4}.$$

Then $f_\tau(x)$ is a meromorphic function on the complex plane. We apply the Poisson summation formula to find that

$$\begin{aligned} S_4(\tau) = 1 + 8\sum_{n=1}^{\infty} \left(\frac{nq^n}{1-q^{2n}} + \frac{q^{2n}}{(1+q^{2n})^2} \right) &= 4\sum_{n\in\mathbb{Z}} f_\tau(n) + \frac{2}{\pi i \tau} = 4\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f_\tau(x) e^{-2\pi i n x} dx + \frac{2}{\pi i \tau} \\ &= 4\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \left(\frac{xe^{\pi i \tau x}}{1-e^{2\pi i \tau x}} + \frac{e^{2\pi i \tau x}}{(1+e^{2\pi i \tau x})^2} \right) e^{-2\pi i n x} dx + \frac{2}{\pi i \tau}. \end{aligned}$$

To evaluate the second integral, we use a rectangular contour as before, with height $-\operatorname{Im}(1/\tau)$ and width R with $R \to \infty$. Note that the only pole surrounded by the contour is $x = -1/2\tau$, which in this case is actually a double pole. The residue is

$$\operatorname{Res}\left(\frac{e^{2\pi i(\tau-n)x}}{(1+e^{2\pi i\tau x})^2}; -\frac{1}{2\tau}\right) = \lim_{x\to-\frac{1}{2\tau}} \frac{d}{dx}\left[\left(\frac{x+\frac{1}{2\tau}}{1+e^{2\pi i\tau x}}\right)^2 e^{2\pi i(\tau-n)x}\right]$$
$$= \frac{1}{2\pi i\tau} e^{\frac{\pi i n}{\tau}} - \frac{2\pi i(\tau-n)}{(2\pi i\tau)^2} e^{\frac{\pi i n}{\tau}} = \frac{n}{2\pi i\tau^2} e^{\frac{\pi i n}{\tau}}.$$

As before, the integrals over the vertical lines $\pm R + ai$ for $0 \le a \le -\operatorname{Im}(1/\tau)$ tend to 0 as $R \to \infty$. By evaluating the integral on the contour from $-\operatorname{Im}(1/\tau) - iR$ to $-\operatorname{Im}(1/\tau) + iR$, and comparing with the real integral, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i\tau x}}{(1+e^{2\pi i\tau x})^2} e^{-2\pi i n x} dx = \frac{1}{\tau^2} \frac{n e^{\frac{\pi i n}{\tau}}}{1-e^{\frac{2\pi i n}{\tau}}} = -\frac{1}{\tau^2} \frac{n e^{-\frac{\pi i n}{\tau}}}{1-e^{-\frac{2\pi i n}{\tau}}}. \tag{11}$$

For the first integral, we choose the rectangular contour through the points $-R$, R, $-R+\operatorname{Im}\frac{2N+1}{2\tau}$, and $R+\operatorname{Im}\frac{2N+1}{2\tau}$ with $R, N \to \infty$. The integrand has poles in the contour at k/τ for k an integer with $1 \le k \le N$, and we again can ignore the integrals over the vertical lines $\pm R + ai$ for $\operatorname{Im}\frac{2N+1}{2\tau} \le a \le 0$ as $R \to \infty$. The integral over the top segment becomes

$$\int_{i\operatorname{Im}\frac{2N+1}{2\tau}-\infty}^{i\operatorname{Im}\frac{2N+1}{2\tau}+\infty} \frac{y e^{\pi i\tau y - 2\pi i n y}}{1-e^{2\pi i\tau y}} dy = \int_{-\infty}^{\infty} -ie^{-\frac{\pi i n(2N+1)}{\tau}} \left(x - \frac{2N+1}{2\tau}\right) \frac{e^{\pi i\tau x - 2\pi i n x}}{1+e^{2\pi i\tau x}} dx.$$

Since $\operatorname{Im}(1/\tau) < 0$, this integral tends to zero as $N \to \infty$. The residues at the poles are

$$\operatorname{Res}\left(\frac{x e^{\pi i\tau x - 2\pi i n x}}{1-e^{2\pi i\tau x}}; \frac{k}{\tau}\right) = \frac{k(-1)^{k+1} e^{-\frac{2\pi i n k}{\tau}}}{2\pi i\tau^2}.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{x e^{\pi i\tau x - 2\pi i n x}}{1-e^{2\pi i\tau x}} dx = \frac{1}{\tau^2} \sum_{k=1}^{\infty} k\left(-e^{-\frac{2\pi i n}{\tau}}\right)^k = -\frac{1}{\tau^2} \frac{e^{-\frac{2\pi i n}{\tau}}}{\left(1+e^{-\frac{2\pi i n}{\tau}}\right)^2}. \tag{12}$$

Combining these, we obtain the desired transformation property

$$S_4(\tau) = -\frac{1}{\tau^2}\left[1 + 8\sum_{n=1}^{\infty}\left(\frac{n e^{-\frac{\pi i n}{\tau}}}{1-e^{-\frac{2\pi i n}{\tau}}} + \frac{e^{-\frac{2\pi i n}{\tau}}}{(1+e^{-\frac{2\pi i n}{\tau}})^2}\right) + \frac{2}{\pi i(-\frac{1}{\tau})}\right] = -\frac{1}{\tau^2} S_4(-\frac{1}{\tau}),$$

as desired. □

We next determine the behavior of the series $S_{2k}(\tau)$ for $\tau \to 1$. We will compare this to the estimate for $\theta^{2k}(\tau)$ as $\tau \to 1$, to show that the ratio $S_{2k}(\tau)/\theta^{2k}(\tau)$ is bounded in this region. We have the following theorem, in this regard. This theorem is stated and proved in [3] in the case $k = 1$.

Theorem 10. [3] *The series $S_{2k}(\tau)$ of Theorem 9 for $k = 1, 2$ satisfies*

$$S_{2k}\left(\frac{\tau-1}{\tau}\right) = (-4i\tau)^k e^{\frac{\pi i\tau k}{2}} + O\left(|\tau|\, e^{\frac{\pi i(k+2)\operatorname{Im}\tau}{2}}\right)$$

as $\operatorname{Im}\tau \to \infty$.

Proof. We follow the proof in [3]. If $\tilde{\tau} := -1/\tau$, and $\tilde{q} := e^{\pi i\tilde{\tau}}$, we have by the Poisson summation formula,

$$S_2(\tilde{\tau}+1) = 2\sum_{n\in\mathbb{Z}} \frac{e^{\pi i n}\tilde{q}^n}{1+\tilde{q}^{2n}} = 2\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{e^{\pi i x + \pi i\tilde{\tau}x - 2\pi i n x}}{1+e^{2\pi i\tilde{\tau}x}} dx = 2\sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{e^{\pi i\tilde{\tau}x - 2\pi i(n-\frac{1}{2})x}}{1+e^{2\pi i\tilde{\tau}x}} dx.$$

By equation (9), we may evaluate the integral to obtain

$$S_2\left(\frac{\tau-1}{\tau}\right) = S_2(\tilde{\tau}+1) = \frac{2i}{\tilde{\tau}} \sum_{n\in\mathbb{Z}} \frac{e^{-\frac{\pi i\left(n-\frac{1}{2}\right)}{\tilde{\tau}}}}{1+e^{-\frac{2\pi i\left(n-\frac{1}{2}\right)}{\tilde{\tau}}}} = -2i\tau \sum_{n\in\mathbb{Z}} \frac{w^{2n-1}}{1+w^{4n-2}} = -4i\tau \sum_{n=1}^{\infty} \frac{w^{2n-1}}{1+w^{4n-2}},$$

where $w := e^{\frac{\pi i \tau}{2}}$. As $\operatorname{Im}\tau \to \infty$, $w \to 0$, and we see that

$$\left|\sum_{n=2}^{\infty} \frac{w^{2n-1}}{1+w^{4n-2}}\right| \leq \frac{|w|^3}{1-|w|}\sum_{n=0}^{\infty}|w|^{2n} \leq \frac{|w|^3}{(1-|w|)(1-|w|^2)} = O(|w|^3).$$

Since the $n=1$ term is $\frac{w}{1+w^2} = w + O(|w|^3)$, we arrive at the desired estimate

$$S_2\left(\frac{\tau-1}{\tau}\right) = -4i\tau e^{\frac{\pi i \tau}{2}} + O(|w|^3|\tau|).$$

Likewise, for the series $S_4(\tau)$, consider the function $\tilde{f}(x)$ analogous to the function $f_\tau(x)$ in (10), defined by

$$\tilde{f}(x) = \frac{xe^{\pi i x + \pi i \tilde{\tau} x}}{1-e^{2\pi i \tilde{\tau} x}} + \frac{e^{2\pi i \tilde{\tau} x}}{(1+e^{2\pi i \tilde{\tau} x})^2}$$

away from $x=0$, and defined at $x=0$ by the limit

$$f(0) := \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{xe^{\pi i x + \pi i \tilde{\tau} x}}{1-e^{2\pi i \tilde{\tau} x}} + \frac{e^{2\pi i \tilde{\tau} x}}{(1+e^{2\pi i \tilde{\tau} x})^2} = -\frac{1}{2\pi i \tau} + \frac{1}{4}.$$

The function $\tilde{f}$ is a meromorphic function on the complex plane. By the Poisson summation formula

$$\begin{aligned} S_4(\tilde{\tau}+1) &= 1 + 8\sum_{n=1}^{\infty}\left(\frac{ne^{\pi i n}\tilde{q}^n}{1-\tilde{q}^{2n}} + \frac{\tilde{q}^{2n}}{(1+\tilde{q}^{2n})^2}\right) = 4\sum_{n\in\mathbb{Z}}\tilde{f}(n) + \frac{2}{\pi i \tilde{\tau}} \\ &= 4\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty}\left(\frac{xe^{\pi i x+\pi i \tilde{\tau} x}}{1-e^{2\pi i \tilde{\tau} x}} + \frac{e^{2\pi i \tilde{\tau} x}}{(1+e^{2\pi i \tilde{\tau} x})^2}\right)e^{-2\pi i n x}dx + \frac{2}{\pi i \tilde{\tau}} \\ &= 4\sum_{n\in\mathbb{Z}}\left(\int_{-\infty}^{\infty}\frac{xe^{\pi i \tilde{\tau} x}}{1-e^{2\pi i \tilde{\tau} x}}e^{-2\pi i\left(n-\frac{1}{2}\right)x}dx + \int_{-\infty}^{\infty}\frac{e^{2\pi i \tilde{\tau} x}}{(1+e^{2\pi i \tilde{\tau} x})^2}e^{-2\pi i n x}dx\right) + \frac{2}{\pi i \tilde{\tau}}. \end{aligned}$$

We can evaluate the integrals using equations (11) and (12) to get

$$\begin{aligned} S_4\left(\frac{\tau-1}{\tau}\right) = S_4(\tilde{\tau}+1) &= -\frac{4}{\tilde{\tau}^2}\sum_{n\in\mathbb{Z}}\left(\frac{e^{-\frac{\pi i(2n-1)}{\tilde{\tau}}}}{\left(1+e^{-\frac{\pi i(2n-1)}{\tilde{\tau}}}\right)^2} + \frac{ne^{-\frac{\pi i n}{\tilde{\tau}}}}{1-e^{-\frac{2\pi i n}{\tilde{\tau}}}}\right) + \frac{2}{\pi i \tilde{\tau}} \\ &= -4\tau^2\left[\sum_{n\in\mathbb{Z}}\left(\frac{e^{\pi i(2n-1)\tau}}{\left(1+e^{\pi i(2n-1)\tau}\right)^2} + \frac{ne^{\pi i n \tau}}{1-e^{2\pi i n \tau}}\right) + \frac{1}{2\pi i \tau}\right] \\ &= -8\tau^2\sum_{n=1}^{\infty}\left(\frac{q^{2n-1}}{(1+q^{2n-1})^2} + \frac{nq^n}{1-q^{2n}}\right), \end{aligned}$$

where $q := e^{\pi i \tau}$, and the summand with $n=0$ is defined by the limit $n \to 0$. As $\operatorname{Im}\tau \to \infty$, $q \to 0$, and

$$\left|\sum_{n=2}^{\infty}\left(\frac{q^{2n-1}}{(1+q^{2n-1})^2} + \frac{nq^n}{1-q^{2n}}\right)\right| \leq \frac{|q|^3}{\left(1-|q|^3\right)^2}\sum_{n=0}^{\infty}|q|^{2n} + \frac{|q|^2}{1-|q|^4}\sum_{n=0}^{\infty}(n+2)|q|^n = O(|q|^2).$$

Since the $n=1$ term is $\frac{q}{(1+q)^2} + \frac{q}{1-q^2} = 2q + O(q^2)$, we obtain the estimate when $k=2$,

$$S_4\left(\frac{\tau-1}{\tau}\right) = -16\tau^2 e^{\pi i \tau} + O(|\tau|^2|q|^2),$$

as desired. □

We now combine the theorems in this section and the previous section to find the following result.

Theorem 11. [3] *For $k = 1, 2, 3$ or 4, and for all $\tau \in \mathbb{H}$, $\theta^{2k}(\tau) = S_{2k}(\tau)$, for S_{2k} the series defined in Theorem 9.*

Proof. We prove this for the cases $k = 1$ and $k = 2$, following the method in [3]. As mentioned in the proof of Theorem 9, proofs of this result for $k = 3$ and 4 can be found in [2], Chapter 9. Let $f(\tau) := S_{2k}(\tau)/\theta_{2k}(\tau)$, this is a holomorphic function on the upper half plane by Theorem 8 (that the theta function is nonzero). By equation (4) and Theorem 9, and because $\theta(\tau)$ and $S_{2k}(\tau)$ are invariant under ϕ^2, we see that $f(\tau)$ is invariant under the action of G, and is thus a modular function. Let $K \subseteq S^2$ denote the compact subset of the Riemann sphere S^2 formed by deleting the appropriate open neighborhoods of 1 and ∞ from $\mathfrak{F} \cup \phi(\mathfrak{F}) \cup (\phi \circ \psi)(\mathfrak{F})$, where $\mathfrak{F}$ is the fundamental domain defined in Section 2. By the extreme value theorem, f is bounded on K, and is bounded on appropriate open neighborhoods of 1 and ∞ by Lemma 7 and Theorem 10; since $\{\alpha(\infty) \mid \alpha = 1, \phi \text{ or } \phi \circ \psi\} = \{1, \infty\}$, f is a bounded modular function. By Lemma 5, G has finite index in Γ, hence by Lemma 4, f is constant. Since $\theta(\tau) \to 1$ and $S_{2k}(\tau) \to 1$ as $\operatorname{Im} \tau \to \infty$, $f(\tau) = 1$, and the desired equality holds. □

5 Sums of 2, 4, 6 or 8 Squares

We now derive the formulas in Theorems 1 and 2. We begin with the equality from equation (2) and Theorem 11,

$$\sum_{n=0}^{\infty} r_{2k}(n) e^{\pi i \tau n} = S_{2k}(n). \tag{13}$$

If $k = 1$, we expand the formula (5) as the power series in $q := e^{\pi i \tau}$, following [3],

$$\begin{aligned} S_2(n) &= 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} = 1 + 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} q^{n(2m-1)} = 1 + 4 \sum_{n=1}^{\infty} \sum_{2d-1|n} (-1)^d q^n \\ &= 1 + \sum_{n=1}^{\infty} \Bigg(\sum_{d|n, d\equiv 1 (\text{mod } 4)} 4 - \sum_{d|n, d \equiv 3 (\text{mod } 4)} 4 \Bigg) q^n. \end{aligned}$$

Comparing both sides of equation (13), we obtain the first formula in Theorem 1. If $k = 2$, we perform a similar expansion for (6) as a power series in q, to get

$$\begin{aligned} S_4(n) &= 1 + 8 \sum_{n=1}^{\infty} \left(\frac{nq^n}{1 - q^{2n}} + \frac{q^{2n}}{(1 + q^{2n})^2} \right) = 1 + 8 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nq^{n(2m-1)} + m(-1)^{m+1} q^{2mn} \\ &= 1 + 8 \sum_{n=1}^{\infty} \Bigg(\sum_{d|n,\, n/d \text{ odd}} d + \sum_{d|n,\, n/d \text{ even}} d(-1)^{d+1} \Bigg) q^n. \end{aligned}$$

Let $n = a \cdot 2^k$ for a odd, then the coefficient of q^n in the power series above is

$$8 \left[\sum_{d|a} d \cdot 2^k + \sum_{d|a} d(1 - 2 - 4 - \ldots - 2^{k-1}) \right] = 8 \sum_{d|a} 3d = 8 \sum_{d|2a} d = 8 \sum_{d|n, 4\nmid d} d,$$

which reproduces the second formula in Theorem 1. The formulas in Theorem 2 can be obtained similarly from equations (7) and (8), see [2] for a derivation in those cases.

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